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Complex variable equations and the numerical solution of harmonic problems for piecewise-homogeneous media☆

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ABSTRACT

Complex variable boundary integral equations are derived using of holomorphicity theorems for plane harmonic problems concerning unit structures with inclusions, pores and lines of discontinuity of the potential and/or the flow. Unlike the method of analytical elements, the equations cover problems in which discontinuities in the potential, flow and conductance can simultaneously be encountered at the contact points. Versions of the equations are given for connected half planes and for periodic and biperiodic problems. Formulae are obtained which determine the effective impedance tensor of the equivalent homogeneous medium in cases when the unit structure is biperiodic or when the representative volume of a structured medium is identified with the basic cell of a biperiodic system. Recurrence quadrature formulae are proposed which enable one to solve the resulting equations effectively using the complex variable boundary element method. They indicate the computational advantages of using the complex variable method compared with the real variable method: the three integrals appearing in the resulting equations are evaluated analytically in the case of linear elements (regular and singular) with the densities approximated using algebraic polynomials of arbitrary degree. In the case of elements (regular and singular) in the form of an arc of a circle, only one integral requires numerical integration when the densities are approximated using complex trigonometrical polynomials of arbitrary degree. It is emphasized that the combination of the linear and curved boundary elements which have been developed enables the smooth part of a contour to be approximated while retaining the continuity of the tangent and avoiding the complications which arise when the smoothness of the approximation of a contour is ensured using conformal mapping. Examples are presented which illustrate the computational merits of the method developed. They show a sharp increase in accuracy (by orders of magnitude) when curved elements are used for the curvilinear parts of a contour and when terminal elements are used to calculate the flow intensity coefficient at singular points (the crack tips the vertices of angular notches and the common vertices of the units of the medium).

A consideration of the internal structure of a medium and the complex processes occurring at the boundaries of the structural elements is an urgent problem in contemporary science, and advances in solving this problem are being stimulated by progress in computer technology and computational methods. It is convenient to use complex variables in the plane problems considered below concerning steady fluid flow, heat transfer and electric current in a piecewise-homogeneous medium when there are discontinuities at the contacts of the structural elements. Their analytical and computational merits in solving plane harmonic and biharmonic problems are well known (for example, see Refs 1–6). Their main merit lies in the possibility of considerably increasing the accuracy of calculations for piecewise-homogeneous media when there are discontinuities in the potential and the flow at the contacts and when there are singularities at the common vertices of the interacting structural elements. This is ensured by the use of highly accurate approximations for which integration in the complex plane enables simple recurrence formulae to be derived, whereas it is extremely difficult to obtain the corresponding formulae in real variables and they look formidable. Compared with the finite element method, the use of complex variable boundary equations is not only favourably characterized by the simplicity in taking account of singular points and discontinuities at contacts but also, and this is especially important. by the possibility of tracking changes in geometry brought about, for example, by the growth of cracks without a basic reconstruction of the mesh.

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However, it should be noted that, as applied to piecewise-homogeneous domains, there is at the present time an incompatibility between the use of complex variables in the numerical solution of biharmonic problems and, it would seem, harmonic problems which are simpler in mathematical respects. Unlike biharmonic problems, for which complex variable boundary integral equations (C-BIE) and the complex variable boundary element method (C-BEM) have been developed in detail and implemented in the form of computer programs (for example, see Refs 6–9), only the method of analytical elements is used in the case of the corresponding harmonic problems,[10–12]. Although it is a version of the indirect C-BEM, the method of analytical elements can only be used when only the potential, only the flow or only the conductance undergoes a discontinuity at a contact. These conditions often occur in problems involving the flow of ground waters, for which this method was developed but, in the general case which is important for problems involving heat transfer and electric charges and for antiplane problems in the theory of elasticity, the potential, flow and conductance can simultaneously undergo a discontinuity. It is therefore advisable to develop a method which is suitable in the general case.

Appropriate C-BIE and C-BEM are given in this paper. It is not a question of an elementary transposition of the results previously obtained for biharmonic problems: unlike these problems in which the densities are complex functions, the densities occurring in the integrals for harmonic problems are real, which leads to complications, in particular in the numerical solution of problems.

1. Formulation of the problem

We will consider a finite or infinite plane piecewise-homogeneous domain which can have pores and lines of discontinuities in the potential and/or the flow. The potential satisfies Laplace's equation in each of the homogeneous subdomains and the flow vector has the components $q_i = k\upsilon$, where $\upsilon_i = \partial U/\partial x_i$ are the components of the potential gradient, x_i are Cartesian coordinates (i = 1,2) and k is the conductance of the medium taken with a minus sign (for brevity, we shall henceforth call k the conductance and its inverse quantity the resistance). Two conditions, which associate the mean values of U_a, q_a and the discontinuities ΔU, Δq of the potential and the normal component of the flow are given at the points of the contacts:

$$\Delta q = F_q(U_a, \Delta U, q_a), \quad \Delta U = F_U(U_a, \Delta q_n, q_a) \tag{1.1}$$

where

$$U_a = \frac{1}{2}(U^+ + U^-), \quad q_a = \frac{1}{2}(q_n^+ + q_n^-), \quad \Delta U = U^+ - U^-, \quad \Delta q_n = q_n^+ - q_n^-$$

A plus (minus) superscript corresponds to the limit of that region with respect to which the normal is external (internal) and F_q and F_u are specified functions or operators. In the particular case of a strongly conducting contact, we have a condition which is known as the "drainage" condition [10] in the mechanics of ground waters:

$$\Delta q_n = k_U \partial U/\partial s, \quad \Delta U = 0$$

where $U = U_a$ is the general value of the potential, $k_U = k_c h$, k_c is the conductance of a thin layer imitating a contact and the derivative is taken along the direction of the tangent to the contact. The "flow" condition

$$\Delta q_n = 0, \quad \Delta U = k_q^{-1} q_a; \quad k_q = k_c/h$$

is satisfied in the opposite case of a weakly conducting contact. For an ideal contact, we have

$$\Delta q_n = 0, \quad \Delta U = 0$$

Conditions (1.1) and their special cases also refer to closed arcs, for example, to cracks or thin inclusions in which the potential and/or the flow can have discontinuities. Conditions of the form (1.1) can also be used for external boundaries if the normal to them is considered to be outward, it is assumed that $q_n^- = 0$, $U^- = 0$ and only one of the two conditions is specified. We shall henceforth use these agreements.

In the case of an infinite domain, we will assume, for brevity, that the potential and the flow are equal to zero at infinity. These two constraints are easily removed. In the case of a finite domain, if the flow is specified at all points of its external boundary, then the overall influx into the domain must be equal to zero. The problem consists of finding a potential which satisfies Laplace's equation at the internal points of the domain and gives a flow which satisfies the specified contact and boundary conditions (1.1).

With the aim of taking advantage of the merits of complex variables, we will formulate this problem by introducing the complex coordinate $z = x_1 + ix_2$ and the complex potential

$$\Omega(z) = kU + iJ; \quad J = J(z) = \int\limits_{s(t_0)}^{s(z)} q_n ds + C \tag{1.2}$$

where $J(z)$ is the stream function, integration is carried out along the length of an arc, starting at an arbitrary point t_0 of a homogeneous subdomain and C is a constant. The functions U and J are harmonic. The function U is single-valued in a homogeneous multiply connected domain while J can acquire a constant on passing around notches and cuts. In order to simplify the account, we shall henceforth adopt the easily removable constraint which excludes multivaluesness of the stream function. The function $\Omega(z)$ is then holomorphic in each of the homogeneous subdomains. Its derivative $\Omega'(z)$ is single-valued regardless of the above-mentioned constraint. The equality

$$\Omega'(z) = i\exp(-i\alpha_z)(q_n - iq_z) \tag{1.3}$$

follows from the Cauchy–Riemann conditions for the derivative $\Omega'(z)$, where α_z is the angle between the arbitrary direction of dz at the point z and the α_1 axis, the normal n is directed to the right from dz and q_n and q_z are the flows in the direction of n and dz respectively.

From relations (1.2) and (1.3), we have

$$U(z) = \mathrm{Re}[k^{-1}\Omega(z)], \quad q_n(z) = \mathrm{Re}[-i\exp(i\alpha_z)\Omega'(z)] \tag{1.4}$$

for the values of the potential and flow in a homogeneous subdomain. It is required to find a piecewise-holomorphic function $\Omega(z)$ such that the limit values of the potential and the flow, determined by formulae (1.4), satisfy the contact and boundary conditions (1.1). We will assume that the function $\Omega(z)$ is equal to zero at infinity in the case of an infinite domain.

2. C-BIE for a homogeneous domain with notches and cuts

We will first consider a finite or infinite homogeneous domain D with p open arcs $L_k(k=1,\ldots,p)$ and m apertures $L_j(j=p+1,\ldots,p+m)$. In the case of a finite domain, we have an additional external contour L_0. We will denote the overall contour by L: its points in D are not included. The direction of circumventing closed contours is chosen such that the domain D remains to the left. The direction of motion along open arcs (cuts) is arbitrary. The initial point in an arc s denoted by a_k and the final point by b_k. To be specific, we will also fix the initial point and final point on each of the closed contours $a_j=b_j(j=p+1,\ldots,p+m)$. The normal n is always directed to the right from the direction of motion. To simplify the account, we will assume that the overall flow through each open and closed contour is equal to zero:

$$\int_{L_k}\Delta q_n ds = 0, \quad k = 0, \ldots, m+p \tag{2.1}$$

where, as above, $\Delta q_n = q_n^+ - q_n^-$.

This constraint only has an effect on the equations containing the stream function and, even in the case of these equations, it is removed by the addition of standard terms with a logarithmic singularity at infinity.[2,6] When conditions (2.1) are satisfied, the stream function is continuous in D and, on the contours L_k, it can be given by the formulae

$$J^{\pm}(t) = J_C^{\pm}(t) + C^{\pm}(t), \quad J_C^{\pm}(t) = \int_{a_k}^{t} q_n^{\pm}(\tau)e^{-i\alpha_\tau}d\tau, \quad k = 0, \ldots, m+p \tag{2.2}$$

$C^{\pm}(t)=C_k^{\pm}$ on L_k and $C_k^{\pm}$ are real constants ($k=0,\ldots,m+p$). For open arcs, we will assume that $C_k^{\pm}=C_k(k=1,\ldots,p)$ and, for closed contours, $q_n^+=q_n$, $q_n^-=0$, $C_k^+=C_k$, $C_k^-=0(p+1,\ldots,p+m)$. The potential must be continuous at the tips of open arcs:

$$\Delta U(a_k) = U^+(a_k) - U^-(a_k) = 0, \quad \Delta U(b_k) = U^+(b_k) - U^-(b_k) = 0, \quad k = 1, \ldots, p \tag{2.3}$$

Under conditions (2.1) and (2.3), the complex potential is a holomorphic function in D. Consequently, it satisfies the equation

$$\frac{1}{2\pi i}\int_L \frac{\Delta\Omega(\tau)}{\tau - z}d\tau = \chi_\Omega(z); \quad \chi_\Omega(z) = \begin{cases}\Omega(z), & z \in D\\ \Omega_a(t), & z = t \in L\\ 0, & z \notin D + L\end{cases} \tag{2.4}$$

The equality (2.4) when $z=t\in L$ expresses the holomorphicity theorem.[6] The equalities when $z\in D$ and $z\notin D+L$ are consequences of the holomorphicity of the function $\Omega(z)$ in D. Substitution of expression (1.2) into Eq. (2.4) gives

$$\frac{1}{2\pi i}\int_L \frac{\Delta(kU) + i\Delta J}{\tau - z}d\tau = \begin{cases}kU(z) + iJ(z), & z \in D\\ (kU)_a + iJ_a, & z = t \in L\\ 0, & z \notin D + L\end{cases} \tag{2.5}$$

Using equality (1.3) in an equation analogous to (2.4), we obtain for the holomorphic function $k^{-1}\Omega'(z)$

$$\frac{1}{2\pi i}e^{i\alpha_z}\int_L \frac{d\Delta U/d\tau + ie^{-i\alpha_\tau}\Delta(q_n/k)}{\tau - z}d\tau = \begin{cases}dU/ds + iq_n(z)/k, & z \in D\\ dU_a/ds + i(q_n/k)_a, & z = t \in L\\ 0, & z \notin D + L\end{cases} \tag{2.6}$$

By definition, in relations (2.4)–(2.6)

$$\Delta\Omega(\tau) = \Omega^+(\tau) - \Omega^-(\tau), \quad \Omega_a(t) = \frac{1}{2}[\Omega^+(t) + \Omega^-(t)], \quad \Delta(kU) = (kU)^+ - (kU)^-$$

$$\Delta J = J^+ - J^-, \quad (kU)_a = \frac{1}{2}[(kU)^+ + (kU)^-], \quad J_a = \frac{1}{2}(J^+ + J^-), \quad \Delta U = U^+ - U^-$$

$$\Delta\left(\frac{q_n}{k}\right) = \left(\frac{q_n}{k}\right)^+ - \left(\frac{q_n}{k}\right)^-, \quad U_a = \frac{1}{2}(U^+ + U^-), \quad \left(\frac{q_n}{k}\right)_a = \frac{1}{2}\left[\left(\frac{q_n}{k}\right)^+ + \left(\frac{q_n}{k}\right)^-\right]$$

As was stipulated, for closed contours

$$U^+ = U, \quad U^- = 0, \quad J^+ = J, \quad J^- = 0, \quad q_n^+ = q_n, \quad q_n^- = 0$$

After separating the real and imaginary parts of equalities (2.5) and (2.6) when $z = t \in L$, they give two pairs of integral equations. Taking one equation from each pair, we obtain a system of two equations which, when supplemented by the two contact conditions (1.1) for the open arcs and the one boundary condition (1.1) for the closed contours, leads to a complete system of equations. The unknown constants in the definition of $J^{\pm}(t)$ (2.2) for open arcs are found from the conditions for the continuity of $J(z)$ in D.

In applications, it is convenient to replace ΔJ by Δq_n in integral (2.5) and the derivative $d\Delta U/dV$ by ΔU in the integral (2.6), using integration by parts. Equality (2.5) then takes the form

$$-\frac{1}{2\pi}\int_L \left[\Delta q_n \ln(\tau - z)ds + i\frac{\Delta(kU)}{\tau - z}d\tau\right] = \begin{cases} kU(z) + iJ(z), & z \in D \\ (kU)_a + iJ_a, & z = t \in L \\ 0, & z \notin D + L \end{cases} \tag{2.7}$$

The real part of equality (2.7) does not contain either the stream function or the derivative of the potential. It only includes quantities appearing in the contact and boundary conditions (1.1). An analogous transformation can be applied to equality (2.6). The resulting equation, like (2.7), holds when the constraints (2.1) are removed. This makes them especially convenient for applications. We therefore write them in the explicit form

$$\operatorname{Re}\left\{-\frac{1}{2\pi}\int_L \left[\Delta q_n \ln(\tau - z)ds + i\frac{\Delta(kU)}{\tau - z}d\tau\right]\right\} = \begin{cases} kU(z), & z \in D \\ (kU)_a, & z = t \in L \\ 0, & z \notin D + L \end{cases} \tag{2.8}$$

$$\operatorname{Re}\left\{-\frac{1}{2\pi}e^{i\alpha_z}\int_L \left[ie^{-i\alpha_\tau}\frac{\Delta(q_n/k)}{\tau - z} + \frac{\Delta U}{(\tau - z)^2}\right]d\tau\right\} = \begin{cases} q_n(z)/k, & z \in D \\ (q_n/k)_a, & z = t \in L \\ 0, & z \notin D + L \end{cases} \tag{2.9}$$

The BIE in equality (2.8) when $z = t \in L$ is singular with an integral in the sense of a principal value (Cauchy). The theory of such integrals and of the corresponding equations is well known.[13] The BIE in equality (2.9) when $z = t \in L$ is hypersingular with a complex finite particular Hadamard integral. The theory of complex finite particular integrals and equations with such integrals exists.[6,7] Both singular and hypersingular integrals are evaluated using efficient recursion formulae along arbitrary curvilinear arcs if the densities are complex functions (for example, see Ref. 6). However, in the equations being discussed, the densities are real functions, which necessitates an appropriate modification of the methods of integration. Such a modification as well as methods for efficiently evaluating integrals with a logarithmic kernel are discussed below. Equations (2.8) and (2.9) when $z \in D$ enable us to calculate the potential and flow within a domain, after the BIE, represented by the second lines, have been solved. The third lines can serve to monitor the accuracy of the calculations.

3. C-BIE for a piecewise-homogeneous domain with inclusions, pores and lines of discontinuity

The initial equation (2.4), the analogous equation for the function $k^{-1}\Omega'(z)$ or any equation following from them can be extended to the case when the domain is not homogeneous and consists of an assembly of homogeneous parts. We will first demonstrate this for the initial equation (2.4). We will assume that there is a system of homogeneous blocks. The matrix, which can enclose the finite blocks, is assumed to be an infinite block. A block within a second unit represents an inclusion and a homogeneous block encompassing an inclusion can be considered as a homogeneous domain with an opening in which the inclusion is inserted. The total number of homogeneous blocks which can have holes and open lines of discontinuity in the potential and/or the flow is equal to n.

We write Eq. (2.4) for each of the homogeneous units

$$\frac{1}{2\pi i}\int_{L_j} \frac{\Delta\Omega_j(\tau)}{\tau - z}d\tau = \chi_{\Omega_j}, \quad j = 1, \ldots, n; \quad \chi_{\Omega_j} = \begin{cases} \Omega_j(z), & z \in D_j \\ \Omega_{ja}(t), & z = t \in L_j \\ 0, & z \notin D_j + L_j \end{cases} \tag{3.1}$$

where

$$\Delta\Omega_j(\tau) = \Omega_j^+(\tau) - \Omega_j^-(\tau), \quad \Omega_{ja}(t) = \frac{1}{2}[\Omega_j^+(t) + \Omega_j^-(t)]$$

D_j is the is the set of internal points in a block j, L_j is the sum of the open and closed contours representing the boundary of the j-th block and the lines of discontinuity in it. Summing equality (3.1) over all block and taking account of the fact that the right-hand sides are equal to zero when $z \notin D_j + L_j$, we arrive at a new equation (2.4), where L is now the *overall* boundary of the system of block (the contact between adjoining blocks is assumed to be a line on which the conductance, potential and flow can have a discontinuity), the direction of motion along a contact is arbitrary and $D = D_1 \cup D_2 \ldots D_n$ is the set of internal points of the system considered. A summation, similar to that used to obtain the new equation (2.4) can be applied to Eqs. (2.8) and (2.9). As a result, we have new Eqs. (2.8) and (2.9) for the system of blocks where L is now the *overall* boundary of the system of blocks $k(z) = k_j$, where k_j is the conductance of the block to which the point z $z \in D_j$ belongs. On the external boundary, $q^- = 0$, $U^- = 0$. When $z = t \in L$, the new equations (2.8) and (2.9), together with the contact and boundary conditions (1.1), give the system of C-BIE for the problem. This system has incidentally been presented by us previously without derivation in a short review[14] discussing results on thermoelasticity. After solving the system, the new equations (2.8) and (2.9), when $z \in D$, determine the potential and flow at the internal points. In the case, when the potential is continuous at all the points of the contacts, the new equation (2.9) is sufficient to solve the problem. In the case when the flow is continuous at the contacts, the new equation (2.9) is sufficient.

Remark. The equations which have been obtained are easily extended to problems involving system of units in connected half-planes. Calculations, analogous to those presented in Ref. 6 for biharmonic problems, can be used for this purpose. Finally, we arrive at equations which only differ from the equations for a whole plane with blocks in that there are additional integrals with kernels which do not have singularities. In particular, for a system of blocks in the lower half-plane ($Re z < 0$), the new equations (2.8) and (2.9) take the form

$$\mathrm{Re}\left\{-\frac{1}{2\pi}\int_L\left[\Delta q_n \ln(\tau - z)ds + i\frac{\Delta(kU)}{\tau - z}d\tau\right] + m\Phi_a(z)\right\} = \begin{cases} k(z)U(z), & z \in D \\ (kU)_a, & z = t \in L \end{cases}$$

$$\mathrm{Re}\left\{-\frac{1}{2\pi}e^{i\alpha_z}\int_L\left[ie^{i\alpha_\tau}\frac{\Delta(q_n/k)}{\tau - z} + \frac{\Delta U}{(\tau - z)^2}\right]d\tau + mV_{an}(z)\right\} = \begin{cases} q_n(z)/k(z), & z \in D \\ (q_n/k)_a, & z = t \in L \end{cases}$$

where $m = (k^{(1)} - k^{(2)})/(k^{(1)} + k^{(2)})$, $k^{(1)}$ is the conductance of the lower half-plane, $k^{(2)}$ is the conductance of the upper half-plane, the supplementary function $\Phi_a(z)$ only differs from the integral term on the left-hand side of equality (2.8) in the replacement of z by its conjugate value $\bar{z}$:

$$\Phi_a(z) = -\frac{1}{2\pi}\int_L\left[\Delta q_n \ln(\tau - \bar{z})ds + i\frac{\Delta(kU)}{\tau - \bar{z}}d\tau\right]$$

and the supplementary function $V_{an}(z)$ differs from the integral term on the left-hand side of equality (2.9) in its sign, the replacement of $e^{i\alpha_z}$ by $e^{-i\alpha_z}$ and the replacement of z by its conjugate value $\bar{z}$:

$$V_{an}(z) = \frac{1}{2\pi}e^{-i\alpha_z}\int_L\left[ie^{-i\alpha_\tau}\frac{\Delta(q_n/k)}{\tau - \bar{z}} + \frac{\Delta U}{(\tau - \bar{z})^2}\right]d\tau$$

A computer program developed for a partitioned system in the whole plane can therefore easily be adapted to solve of problems involving systems of units in connected half-planes.

4. C-BIE for biperiodic problems

We will first consider an infinite plane with a biperiodic system of open and closed contours. Such a system with one open and one closed contour in each of the cells is shown in Fig. 1. The results will then be extended to biperiodic systems of units with pores, inclusions and lines of discontinuity. We will denote the corresponding domain by D. The periods are given by the complex vectors $2\omega_1$ and $2\omega_2$. The condition for their non-collinearity is expressed by the inequality $Im(\bar{\omega}_1\omega_2) \neq 0$. To be specific, we will assume that the angle $\alpha_{\omega 2}$ between the periods is measured from $2\omega_1$ to $2\omega_2$ and that it is positive and does not exceed π. The number $Im(\bar{\omega}_1\omega_2)$ is then positive and is equal to $S/4$, where S is the area of a basic cell.

We will place the origin of the coordinates in the basic parallelogram $ABCD$ outside the contour L, which includes p open arcs $L_k (k = 1, \ldots, p)$ and m closed contours $L_j (j = p + 1, \ldots, p + m)$, corresponding to the cuts. The boundary conditions for the congruent contours in the other cells repeat the conditions for the contour of the basic cell. Consequently, the derivative $\Omega'(z)$ of the complex potential is a biperiodic function: $\Omega'(z + k2\omega_j) = \Omega'(z)$ for any integer k ($j = 1,2$). The function $\Omega(z)$, being an integral of $\Omega'(z)$, is quasiperiodic: $\Omega(z + 2\omega_j) = \Omega(z) + 2\beta_j$, where $2\beta_j$ is a cyclic constant in the direction of $2\omega_j (j = 1, 2)$. For the potential and the stream function, we have

$$kU(z + 2\omega_j) = kU(z) + k2\delta_j, \quad J(z + 2\omega_j) = J(z) + 2\gamma_j \tag{4.1}$$

where $k2\delta_j = Re2\beta_j$, $2\gamma_j = Im2\beta_j$ are real cyclic constants and k is the conductance of the plane.

As above, for a start we take conditions (2.1), which express the fact that the influx into each of the apertures and into each open arc is equal to zero. The stream function is then single-valued in D and is given by formula (2.2) on the contours L_k. Consequently, the complex potential $\Omega(z)$ is a quasiperiodic holomorphic function in the domain D.

A solution will be sought which gives a continuous potential, that is, conditions (2.3) at the tips of the open arcs must be satisfied in the basic cell. With these conditions, the holomorphicity theorem for quasiperiodic functions [6] gives the equations

$$\frac{1}{2\pi i}\int_L \Delta\Omega(\tau)[\varsigma(\tau - z) - \varsigma(\tau)]d\tau - C_\eta z = \chi_\Omega(z); \quad C_\eta = \frac{2}{\pi i}(\beta_1\eta_2 - \beta_2\eta_1) \tag{4.2}$$

$$\frac{1}{2\pi i}\int_L \Delta\Omega(\tau)d\tau = -C_\omega \tag{4.3}$$

The functions $\Delta\Omega(\tau)$ and $\chi_\Omega(z)$ are defined in the same way as in Section 2, $\varsigma(z)$ is the Weierstrass zeta-function with periods $2\omega_1$ and $2\omega_2$ and $2\eta_1$ and $2\eta_2$ are the cyclic constants of the zeta-function in the direction of the periods $2\omega_1$ and $2\omega_2$ respectively. These constants can be found using the formula $2\eta_j = 2\varsigma(\omega_j)$ ($j = 1, 2$). As usual, for closed contours we take $\Omega^+(t) = \Omega(t)$, $\Omega^-(t) = 0$.

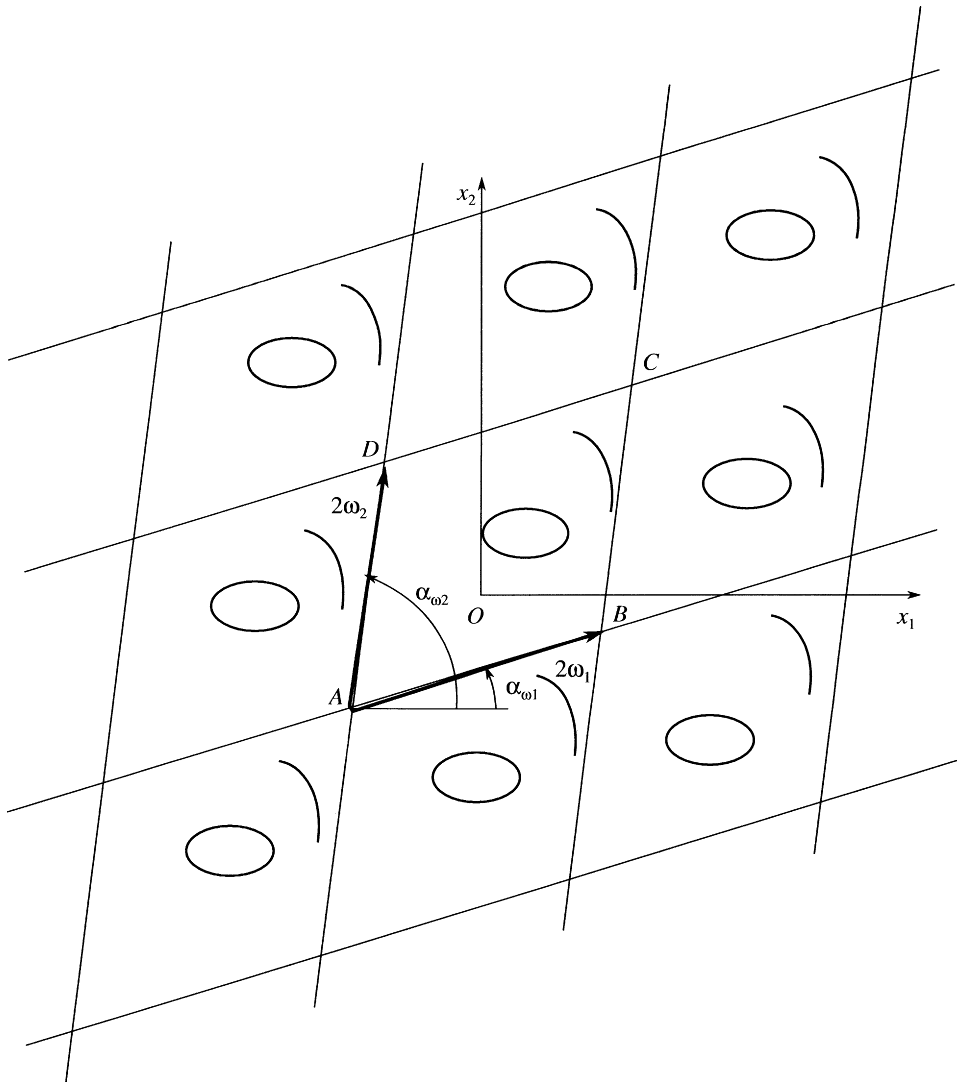

Fig. 1.

The Legendre identity $\eta_1\omega_2 - \eta_2\omega_1 = \pi i/2$ enables us to express C_η in terms of C_ω:

$$C_\eta = C_\omega \frac{\eta_1}{\omega_1} - \frac{\beta_1}{\omega_1}$$

Equation (4.3) can then be written in the form

$$C_\eta = -\frac{\eta_1}{\omega_1}\frac{1}{2\pi i}\int_L \Delta\Omega(\tau)d\tau - \frac{\beta_1}{\omega_1} \tag{4.4}$$

that is, the left-hand side of equality (4.2) actually only contains one complex constant $\beta_1 = k\delta_1 + i\gamma_1$ of the function $\Omega(z)$. The two real constants δ_1 and γ_1 have to be specified. Below, when discussing the averaging of properties, we will express them in terms of physical quantities-mean flows. Equation (4.3) is therefore satisfied by specifying C_η using formula (4.4) and we can concentrate on Eqs. (4.2).

We now consider a biperiodic block system with the same periods $2\omega_1$ and $2\omega_2$ which consists of blocks which are inserted into apertures or outside apertures in a homogeneous plane with excisions and cuts. The initial blocks themselves can have inclusions, pores and open arcs of discontinuity in the potential and/or the flow. Using appropriate cuts, these blocks can be represented as the sum of simply connected homogeneous subdomains D_j with contours L_j in the basic block. These subdomains are repeated with periods $2\omega_1$ and $2\omega_2$ in the other cells.

For a finite subdomain D_j in the basic cell, formula (3.1) can be written in the form

$$\frac{1}{2\pi i}\int_{L_j} \Delta\Omega_j(\tau)[\varsigma(\tau - z) - \varsigma(\tau)]d\tau = \chi_{\Omega_j}(z) \tag{4.5}$$

where it has been assumed that $\Omega_j^+(t) = \Omega_j(t)$, $\Omega_j^-(t) = 0$. The equality (4.5) is obtained from equality (3.1) if account is taken of the definition of the zeta-function

$$\varsigma(z) = \frac{1}{z} + \sum_{k_1,k_2}{}' \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2}\right), \quad w = n_1 2\omega_1 + n_2 2\omega_2$$

(summation is carried out over all positive, negative and zero pairs of m_1 and m_2 with the exception of $m_1 = m_2 = 0$. Actually, if z belongs to the basic cell, the point z - w lies outside it and, consequently, outside D_j. Then, according to equality (3.1) when $z \notin D_j + L_j$, the corresponding integral on the left-hand side of equality (4.5) is equal to zero. The integrals of the terms $1/w$ and z/w^2 are equal to zero since the function $\Omega_j(z)$ is holomorphic in D_j. Consequently, the replacement of $1/(\tau - z)$ by $\varsigma(\tau - z)$ does not change the integral on the left-hand side of equality (3.1). As far as the term with $\varsigma(\tau)$ is concerned, a similar argument gives

$$\int_{L_j} \Delta\Omega_j(\tau)\varsigma(\tau)d\tau = \int_{L_j} \frac{\Delta\Omega_j(\tau)}{\tau} d\tau = \Omega_j(0)$$

Hence, when the point z=0 does not belong to the j-th block, this integral is equal to zero. Otherwise, it is possible to put $\Omega_j(0) = 0$ since the complex potential is defined apart from an arbitrary constant. Finally, the replacement of $1/(\tau - z)$ by $\varsigma(\tau - z) - \varsigma(\tau)$ does not change the integral on the left-hand side of equality (3.1) and, in the case of homogeneous units, Eq. (3.1) can therefore be written in the form (4.5).

Equation (4.5) has the same form as (4.2). The equations can therefore be summed over the homogeneous matrix and over all the homogeneous blocks. The same can be done for any other group of equations following from (4.5) and (4.2), after separating the real and imaginary parts and/or integrating by parts. Finally, application of the same procedure that led to the new equations (2.8) and (2.9) to Eqs. (4.2) and (4.5) gives

$$\mathrm{Re}\left\{-\frac{1}{2\pi}\int_L \left[\Delta q_n \ln\frac{\sigma(\tau - z)}{\sigma(\tau)} ds + i\Delta(kU)[\varsigma(\tau - z) - \varsigma(\tau)]d\tau\right] - C_\eta z\right\} = \begin{cases} kU(z), & z \in D \\ (kU)_a, & z = t \in L \end{cases} \tag{4.6}$$

$$\mathrm{Re}\left\{-\frac{1}{2\pi}e^{i\alpha_z}\int_L [ie^{-i\alpha_\tau}\Delta(q_n/k)\varsigma(\tau - z) + \Delta U \wp(\tau - z)]d\tau + ie^{i\alpha_z}C_\eta\right\} = \begin{cases} q_n(z)/k, & z \in D \\ (q_n/k)_a, & z = t \in L \end{cases} \tag{4.7}$$

where $\sigma(z)$ is the Weierstrass sigma-function, $\wp(z) = -d\varsigma/dz$ is the Weierstrass gamma-function, L is the overall boundary of the system of blocks in the basic cell, and $D = D_1 \cup D_2 \ldots \cup D_n$ is the set of internal points of the system being considered in the basic cell. Artificial cuts made in the initial blocks with inclusions, pores and cracks drop out from L since there are no discontinuities along them: $\Delta q_n = 0$, $\Delta U = 0$ and $\Delta k = 0$.

Using the series presented above for $\varsigma(z)$ and the series

$$\ln\sigma(z) = \ln z + \sum_{k_1,k_2}{}' \left[\ln\left(1 - \frac{z}{w}\right) + \frac{z}{w} + \frac{1}{2}\frac{z^2}{w^2}\right], \quad \wp(z) = \frac{1}{z^2} + \sum_{k_1,k_2}{}' \left[\frac{1}{(z-w)^2} - \frac{1}{w^2}\right]$$

it is possible to adapt a computer program intended for non-periodic problems on systems of blocks to solve analogous biperiodic problems.

Remark. Similar arguments give the C-BIE for periodic problems with period π. The resulting equations reproduce the equations obtained in Section 3 with the logarithmic kernel $\ln(\tau - z)$ replaced by $\ln[\sin(\tau - z)]$, the singular kernel $1/(\tau - z)$ by $\mathrm{ctg}(\tau - z)$ and the hypersingular kernel $1/(\tau - z)^2$ by $1/\sin^2(\tau - z)$.

5. Equations for the effective properties

The equations of the preceding section enable us to solve the averaging problem, that is, to replace an inhomogeneous medium with pores, inclusions, cracks and other lines of discontinuity with a macroscopically equivalent homogeneous medium. Equivalence means that the mean potential gradients are the same in the actual and homogeneous media for the same mean flows.

The mean potential gradient. We will now consider a continuous plane (without units, cuts and excisions). Suppose a homogeneous gradient field $\upsilon^\infty = \upsilon_1^\infty + i\upsilon_2^\infty$ is specified in this plane. This field is periodic in the case of an arbitrary period and the corresponding potential $U^\infty = \upsilon_1^\infty x_1 + \upsilon_2^\infty x_2$ is therefore a quasiperiodic function for any periods. For an element of length ds, taken in a direction at an angle α_z, we obtain

$$dU^\infty/ds = \upsilon_1^\infty \cos\alpha_z + \upsilon_2^\infty \sin\alpha_z$$

In particular, in the direction of the period $2\omega_j$, we have

$$dU^\infty/ds_j = \upsilon_1^\infty \cos\alpha_{\omega j} + \upsilon_2^\infty \sin\alpha_{\omega j}, \quad j = 1, 2$$

For an increment U^∞ along the period $2\omega_j$ we obtain from this that

$$U^\infty(z + 2\omega_j) - U^\infty(z) = (\upsilon_1^\infty \cos\alpha_{\omega j} + \upsilon_2^\infty \sin\alpha_{\omega j})|2\omega_j| = 2\mathrm{Re}(\upsilon^\infty \overline{\omega}_j)$$

Consequently, the cyclic constants of the potential $U^\infty(z)$ are equal to $2Re(\upsilon^\infty\bar{\omega}_j)$. They will be the same as the cyclic constants determined by the first of formulae (4.1) for a system of biperiodic blocks if $\sigma_j = Re(\upsilon^\infty\bar{\omega}_j)(j = 1, 2)$. Hence, the cyclic constants of the potential for a plane with units, pores, inclusions and lines of discontinuity are expressed in terms of the mean gradient of the linear potential $U^\infty(z)$ of a continuous plane (without an internal structure).

The mean flow. We will now calculate the increment in the stream function along the sides AB and AD of the basic cell (Fig. 1). We have

$$J_{AB} = J(z_B) - J(z_A) = 2\gamma_1 = \int\limits_{AB} q_n ds, \quad J_{AD} = J(z_D) - J(z_A) = 2\gamma_2 = \int\limits_{AD} q_n ds$$

The mean flows for these increments are $q_{aj} = 2\gamma_j/|2\omega_j|$.

We will now consider the homogeneous (effective) flow field $q^\infty = q_1^\infty + iq_2^\infty$ in the continuous plane. It is periodic for any period. In the local (n, τ) system with the τ axis in a direction which makes an angle α_τ with the x_1 axis, we have

$$q_n^\infty + iq_\tau^\infty = i\exp(-i\alpha_\tau)q^\infty$$

Consequently,

$$q_n^\infty = \mathrm{Re}[i\exp(-i\alpha_\tau)q^\infty] = -\mathrm{Im}[q^\infty\exp(-i\alpha_\tau)]$$

It follows from this that, for the directions of the periods,

$$q_{nj}^\infty = -\mathrm{Im}[q^\infty\exp(-i\alpha_{\omega j})]$$

Then, since $q_{aj} = 2\gamma_j/|2\omega_j|$, we obtain that the actual mean flow q_{aj} along a period of $2\omega_j$ will be equal to the effective flow q_{nj}^∞ along this direction when $\gamma_j = -Im(q^\infty\bar{\omega}_j)(j = 1, 2)$. Hence, the cyclic constants of the stream function of the initial problem are expressed in terms of the mean flow in a continuous plane (without internal structure).

The constants β_j, C_ω and C_η. The formula

$$\delta_j = \mathrm{Re}(\upsilon^\infty\overline{\omega_j}), \quad \gamma_j = -\mathrm{Im}(q^\infty\overline{\omega_j}) \tag{5.1}$$

where k_0 is the conductance of the matrix, which expresses the cyclic constants $2\beta_j$ in terms of the mean values of the gradient and the flow, follows from the equalities

$$2\beta_j = k_0 2\mathrm{Re}(\upsilon^\infty\overline{\omega_j}) - 2i\mathrm{Im}(q^\infty\overline{\omega_j}), \quad j = 1, 2$$

Substitution of expression (5.1) into the expression for C_ω gives, after some reduction,

$$C_\omega = \frac{S}{2\pi}(k_0\upsilon^\infty - q^\infty)$$

where S is the area of the basic cell. Using expression (5.1) in equality (4.3), we obtain the fundamental relation between the effective gradient and the effective flow

$$\upsilon^\infty = r_0\frac{i}{S}\int\limits_L \Delta\Omega(\tau)d\tau + r_0 q^\infty \tag{5.2}$$

where $r_0 = 1/q_0$ is the resistance of the matrix enclosing the blocks.

Substitution of expression (5.2) into the equality (5.1) gives the required expression for the complex cyclic constants β_j solely in terms of the two real constants q_1^∞ and q_2^∞, which appear in the definition $q^\infty = q_1^\infty + iq_2^\infty$ of the effective flow:

$$\beta_j = -\frac{1}{S}\mathrm{Im}\left(\overline{\omega_j}\int\limits_L \Delta\Omega(\tau)d\tau\right) + \omega_j\overline{q^\infty}, \quad j = 1, 2 \tag{5.3}$$

Using this equality, the constant C_η, defined by formula (4.4), can be expressed in terms of the effective flow $q^\infty = q_1^\infty + iq_2^\infty$:

$$C_\eta = -\frac{\eta_1}{\omega_1}\frac{1}{2\pi i}\int\limits_L \Delta\Omega(\tau)d\tau + \frac{1}{S\omega_1}\mathrm{Im}\left(\overline{\omega_j}\int\limits_L \Delta\Omega(\tau)d\tau\right) - \overline{q^\infty} \tag{5.4}$$

Note that the integral in equalities (5.2)–(5.4), after using the definition $\Omega(z) = k(z)U(z) + iJ(z)$ and integrating the term $J(z)$ by parts, takes the form

$$\int\limits_L \Delta\Omega(\tau)d\tau = \int\limits_L [\Delta(kU) - ie^{-i\alpha_\tau}\tau\Delta q_n]d\tau \tag{5.5}$$

Finally, taking account of relations (5.4) and (5.5), we conclude that Eqs. (4.6) and (4.7) turn out to have been formulated in terms of only the potential and the flow and only contain two real constants: the specified mean flows q_1^∞ and q_2^∞ along the coordinate axes.

The effective resistance tensor. The effective resistance tensor is found by solving two problems with the homogeneous contact and boundary conditions: 1) $q_1^\infty = 1$, $q_2^\infty = 0$ and 2) $q_1^\infty = 0$, $q_2^\infty = 1$. The solution of each of these gives the functions $\Delta(kU)$ and Δq determining the integral on the right-hand side of equality (5.5) which, when substituted into relation (5.2), gives the mean gradient $\upsilon^\infty = \upsilon_1^\infty + i\upsilon_2^\infty$. We denote the values of υ_1^∞, υ_2^∞ obtained from relation (5.2) for the solution of the first problem by $r_{11} = r_{a11} + r_0$, $r_{12} = r_{a12}$ and the values

$\upsilon_1^\infty, \upsilon_2^\infty$ obtained from relation (5.2) for the solution of the second problem by $r_{21} = ra_{21}, +r_0$. Then, in the case of an arbitrary mean flow, we have

$$\left\| \begin{matrix} \upsilon_1^\infty \\ \upsilon_2^\infty \end{matrix} \right\| = \boldsymbol{R} \left\| \begin{matrix} q_1^\infty \\ q_2^\infty \end{matrix} \right\|, \quad \boldsymbol{R} = \left\| \begin{matrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{matrix} \right\| = r_0 \boldsymbol{I} + \boldsymbol{R_a}, \quad \boldsymbol{I} = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \quad \boldsymbol{R_a} = \left\| \begin{matrix} r_{a11} & r_{a12} \\ r_{a21} & r_{a22} \end{matrix} \right\|$$

where $\boldsymbol{R}$ is the effective resistance matrix. It is seen that the effective resistance matrix $\boldsymbol{R}$ is the sum of a spherical tensor $r_0\boldsymbol{I}$ for a structureless plane and the tensor $\boldsymbol{R}_a$ of the additional resistances which are induced by the existence of structural elements. This makes it more convenient for applications than the effective conductance tensor which is inverse to it. This latter tensor can only be considered as being additive in cases when the additional resistances r_{akj} are small in absolute magnitude compared with $|r_0|$.

6. Numerical solution of the C-BIE using the C-BEM

It was mentioned in the introduction that, when the density is a complex function, the complex variable singular and hypersingular integrals can be evaluated for arbitrary curvilinear elements using efficient recursion formulae which have been presented earlier.[6,7] This advantage of complex variables is partially lost in the case of Laplace's equation when the density turns out to be a real function. The point is that now its approximation, by complex polynomials, for example, is not a real function for an arbitrary curvilinear element.

This difficulty can be overcome in two ways. The first consists of mapping the arc of integration onto a real segment and using a real approximation of the density in it in the form of real polynomials, for example. This approach was used systematically in Refs 10–12. Its drawback lies in the need to take account of the pole of the mapped function for each type of elements. On the other hand, the second method, which is described below, uses integration in complex variables but the element of integration and the approximation of the density in it must be such that the approximating function turns out to be real at the points of the element. The drawback of this approach lies in the restricted class of boundary elements for which such an approximation is fairly simply accomplished. However, it does include two important types of boundary elements: 1) linear elements and 2) elements along arcs of circles. It is clear that these two forms of elements enable an arbitrary contour to be fairly accurately represented by a set of them. We emphasize that the use of elements in the form of arcs of circles on smooth parts of a contour ensures an approximation with a continuous tangent. In order to simplify the quadrature formulae, the elements are converted to a standard form by a linear transformation of the coordinates.

A rectilinear element with its start at the point b and end at the point c is transformed in the interval [1,−1] using the complex coordinate $z' = x_1' + ix_2'$, which is related to the initial coordinate $z = x_1 + ix_2$ by the linear equation $z = z_c + lz' \exp(i\alpha_c)$, where $z_c = (b + c)/2$ is the middle point of the element, $l = |b + c|/2$ is its half length and α_c is the angle which the element makes with the x_1 axis. It is important that, at the points of the element, $\tau' = \tau' = x_1'$.

An element along an arc of a circle starting at the point b and ending at the point c with the centre of the circle at the point z_c is transformed into an arc of unit radius starting at the point $\overline{\tau_0}$ and ending at the point τ_0, with its centre at the new origin of coordinates using a new complex variable $z' = x_1' + ix_2'$, which is related to the initial variable by the linear equation $z = z_c + lz' \exp(i\beta_c)$, where $l = |b - z_c| = |c - z_c|$ is the radius of the initial arc, β_c is the angle of the normal at the middle point of the arc to the x_1 axis with

$$2\beta_c = \arg b + \arg c, \quad \overline{\tau_0} = \exp(-i\theta_0), \quad \tau_0 = \exp(i\theta_0), \quad 2\theta_0 = \arg c - \arg b$$

and $2\theta_0$ is the aperture angle of the arc. It is important that $\tau' = 1/\tau'$ at the points of an arc element.

The function f, which is defined at the points of an element (b,c), can be approximated on the corresponding standard element using the basis functions $B_k(\tau')$ with n mesh points $\tau_j'(k, j = 1, \ldots, n)$ which are such that $B_k(\tau_j') = \delta_{kj}$ (δ_{kj} is the Kronecker delta). Then,

$$f(\tau') = \sum_{k=1}^{n} f_k B_k(\tau') \tag{6.1}$$

where $f_k = f(\tau_k') = f(\tau'(\tau_k))$ is the value of the function at the k-th mesh point (k = 1,. . .,n). The functions f_k are real in the harmonic problems being considered. In view of the fact that $B_k(\tau_j') = \delta_{kj}$, the right-hand side of equality (6.1) is real at the mesh points τ_k'. However, it cannot be real at the other points of the transformed element if the basis functions are not real at these points. It is therefore necessary to focus on basis functions for which the right-hand side of equality (6.1) is real at the points of the transformed element.

In the case of a straight element since $\overline{\tau'} = \tau' = x_1'$, any real function of the real argument τ' can be used. Algebraic polynomials are the simplest basis in [-1,1]. Then, $B_k(\tau')$ are Lagrange polynomials $P_k(\tau')$:

$$B_k(\tau') = P_k(\tau') = \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{\tau' - \tau_j'}{\tau_k' - \tau_j'} = \sum_{s=0}^{n} c_{ks}\tau'^s, \quad k = 1, \ldots, n \tag{6.2}$$

where c_{ks} are constants which are determined by the arrangement of the mesh points.

In the case of an element along the arc of a circle the choice of the basis functions with the necessary property is not so obvious, since the coordinate $\tau' = e^{i\theta}$ in it is not real. Nevertheless, bearing in mind that trigonometric polynomials are a simple basis in $[-\theta_0, \theta_0]$ it is advisable to use them for the approximations

$$B_k(\tau') = \tilde{P}_k(\tau') = \sum_{s=-m}^{m} \tilde{c}_{ks}\tau'^s, \quad k = 1, \ldots, n \tag{6.3}$$

where $n = 2m + 1$ is an odd number and, in the general case, $\tilde{c}_{ks}$ are complex constants, which are determined by the arrangement of the points τ'_k on the arc of the circle. They are easily calculated using the relation between the coefficients c_{kj} and Lagrange polynomials of degree $2m$ (see Ref. 6):

$$\tilde{c}_{ks} = (\tau'_k)^m c_{kj}, \quad k = 1, \ldots, n, \quad s = -m, \ldots, m, \quad j = s + m$$

At least some of the coefficients $\tilde{c}_{ks}$ are not real. Nevertheless, it is easily verified, by taking the conjugate in equality (6.3) and taking account of the fact that $\overline{\tau'} = 1/\tau'$, that the basis functions $B_k(\tau')$ are real in an arc of the unit circle.

In order to evaluate the integrals appearing in the C-BIE obtained above, the integration contour is represented by a set of curved and rectilinear elements, each of which is transformed to the standard form by a linear transformation and the density is approximated according to relation (6.2) in the case of a standard rectilinear element or, according to relation (6.3), for a standard curved element. We finally arrive at standard types of integrals over standard elements. It is clear from Eqs (2.8) and (2.9) that it is only necessary to have quadratures for three types of integrals: with a logarithmic, singular and hypersingular kernel. In the case of approximations (6.2) and (6.3), they have been given earlier[6,7] for the singular and hypersingular integrals in the form of analytical recursion formulae. Actually, the two subroutines for a standard rectilinear element and two subroutines for a standard curved element, developed for biharmonic problems, can be used in the problems being considered without any modifications. Integrals with a logarithmic kernel are transformed to singular and hypersingular integrals by integration by parts but the starting integral has to be found numerically in the case of a curved element. The four subroutines, which are already used in solving biharmonic problems, are supplemented with a subroutine for calculating the starting logarithmic integral in the case of a curved element.

Finite elements. The case when the density or its derivative becomes infinite at a certain point of an element requires special attention. To be specific, we will assume that this point c is the end of an element and that the asymptotic behaviour of the density has the form d^{-a}, where $a = n_1/m_1$ is a rational fraction ($n_1 < m_1$, where n_1 and m_1 are positive integers), and $d = |c - \tau|$ is the distance from the point τ to the end point c. The linear transformations $z = z_c + lz' \exp(i\alpha_c)$ and $z = z_c + lz' \exp(i\beta_c)$ do not change the form of a singularity since, in the new coordinates, we have $d = ld'$, where $d' = |c' - \tau'|$. Here, $c' = 1$ for a rectilinear element and $c' = \tau_0$ for a curved element. It is therefore sufficient to consider density approximations for standard elements. We will denote the real density in the transformed terminal element by $g(\tau')$. Taking account of its asymptotic behaviour, we have $g(\tau') = (d')^{-a} f(\tau')$, where $f(\tau')$ is a smooth real function in the element. Approximation (6.1) can be used for $f(\tau')$. Approximation of $g(\tau')$ then takes the form

$$g(\tau') = (d')^{-a} \sum_{k=1}^{n} f_k B_k(\tau')$$

In the case of a rectilinear element $[-1, 1]$, the difference $1 - \tau'$ is a real positive number and, therefore, $d^{-a} = (1 - \tau')^{-n_1/m_1}$. The right-hand side of the last expression is a holomorphic function of the complex argument z' in a plane with a cut from the point + 1 to minus infinity. The product $(\tau')^s (1 - \tau')^{-n_1/m_1}$ is the same. In the case of the functions $B_k(\tau')$, taken as the Lagrange polynomials (6.2), we therefore have the approximation

$$g(\tau') = (1 - \tau')^{-n_1/m_1} \sum_{k=1}^{n} f_k P_k(\tau')$$

The singular and hypersingular integrals in this approximation are easily evaluated for any integers n_1 and m_1 using recursion formulae [15] (formulae have been given earlier[6,7] for an ordinary root singularity). An integral with a logarithmic kernel is reduced to a singular integral. Hence, no problems arise in taking account of the singular behaviour of the density in the case of a rectilinear element.

In the case of a curved element, $d' = |c' - \tau'|$ cannot be replaced by $c' - \tau'$ since the difference $c' - \tau'$ is not real in such an element. This difficulty can be overcome by using $Re(\tau_0 - \tau')^{-a}$ instead of $|\tau_0 - \tau'|^{-a}$. In the limit when $\tau' \to \tau_0$, these functions only differ by the factor $\cos(a\pi/2)$ which can be included in the function $f(\tau')$. The asymptotic behaviour is then described by the formula

$$g(\tau') = f(\tau')\mathrm{Re}(\tau_0 - \tau')^{-a} \tag{6.4}$$

We now note that, in the transformed variables, any of the integrals in the C-BIE which have been obtained can be written in the form

$$\mathrm{Int} = \mathrm{Re}\int_{L_e} g(\tau')K(\tau' - z')d\tau' = \int_{L_e} g(\tau')\mathrm{Re}[K(\tau' - z')d\tau'] \tag{6.5}$$

since the density $g(\tau')$ is real. After substituting expression (6.4) into integral (6.5), we then obtain

$$\mathrm{Int} = \frac{1}{2}\mathrm{Re}\int_{L_e} f(\tau')(\tau_0 - \tau')^{-a}[K(\tau' - z')d\tau' + K(\overline{\tau'} - \overline{z'})d\overline{\tau'}] \tag{6.6}$$

The symbol for the real part appears outside the integral sign. The approximation (6.1) with the basis function (6.3) is now already applicable. Since, for the transformed, arc we have $\tau' = 1/\tau'$, no difficulties are encountered in evaluating the singular and hypersingular integrals on the right-hand side of equality (6.6). Again, the quadratures obtained for problems in the theory of elasticity[6,7,15] can be used. For integrals with a logarithmic kernel, once again just one starting integral has to be evaluated numerically and, then, only in the case of a curved element. Having subroutines for evaluating the integrals for standard elements, it is easy to form the matrix of the weighting factors of the C-BIE for any block system.

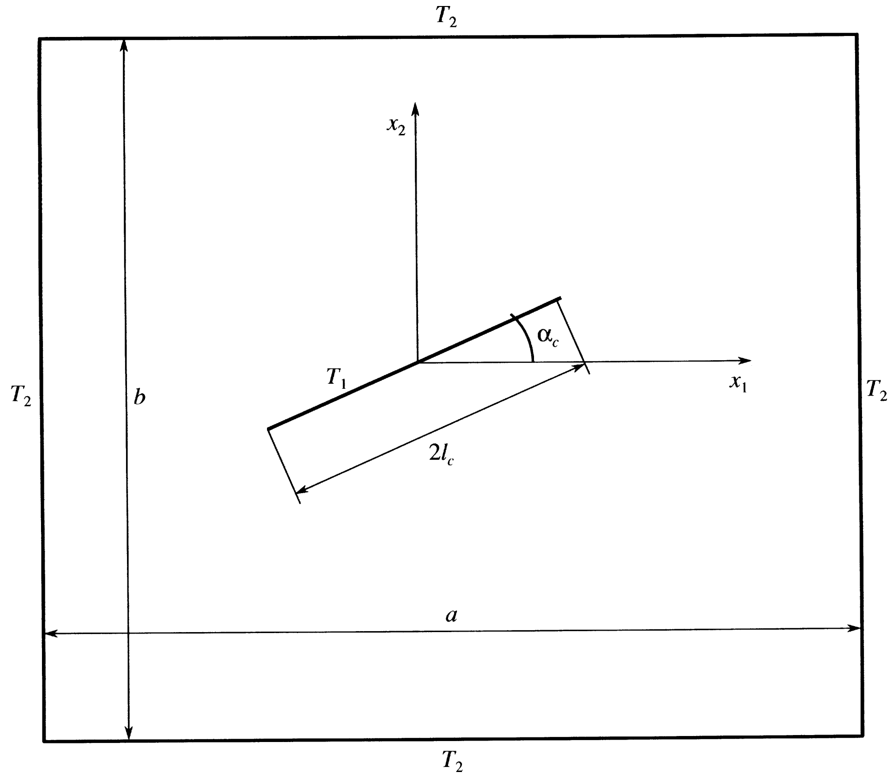

Fig. 2.

7. Numerical experiments and examples

We will first illustrate the increase in accuracy when, first, curved elements are used for the peripheries and, second, terminal elements are used for a fissure when the flow at its tips tend to infinity. To be specific, we will consider thermal problems.

The calculations were carried out with double precision using three-point boundary elements (rectilinear, curved, regular and terminal), that is, $n=3$ in equalities (6.2) and $m=1$ in equalities (6.3). All the integrals, apart from the starting integral for an arc of a circle in the case of a logarithmic kernel, were evaluated using quadrature formulae and subroutines developed for biharmonic problems (see Ref. 6 for example). The starting integral with a logarithmic kernel was evaluated along a curved element when $s=0$ in approximation (6.3) using Gauss' quadrature formula with ten nodes. The logarithmic singularity was isolated for the collocation points belonging to the arc of integration and integrated analytically so that the Gauss' formula was always applied to a function without a singularity. A personal computer with a clock frequency of 2 MHz and a core memory of 512 Mb was used. The computation time did not exceed 30 seconds in the case of calculations for the finest mesh of the elements described below.

Example 1. Increase in accuracy using curved elements. Consider a tube with an inner radius R_1, outer radius R_2, conductance k, temperature T_1 on the inner boundary and T_2 on the outer boundary. The exact solution, with which the numerical results can be compared, for a flow in a direction opposite to the radius r, is given by the formula

$$q_{-r} = -k/r \cdot (T_2 - T_1)/\ln(R_2/R_1)$$

It was assumed in the calculations that

$$T_1 = 0, \quad T_2 = 1, \quad k = -1, \quad R_1 = 1$$

In order for the lengths of the boundary elements on the inner and outer boundaries to be substantially different, it was assumed that $R_2/R_1 = 10$ and 100. Then, the flow to ten correct digits in the numerator is $q_{-r} = 0.43429448190/r$ for $R_2/R_1 = 10$ and, for $R_2/R_1 = 100$, the numerator is half as large. In the example being discussed, the sole source of error (apart from rounding off errors) can only be the error in the numerical evaluation of the starting integral in the case of a logarithmic kernel.

Calculations using the C-BEM program in the case of six curved elements, three on the inner and three on the outer contour, gave seven correct digits both for $R_2/R_1 = 10$ and for $R_2/R_1 = 100$. This is actually a rounding off error. Experiments using rectilinear elements instead of curved elements showed that, in the case of 32+32 = 64 elements, there is an error in the second digit, for 64+64 = 128 elements there is an error in the third digit and for 128 + 128 = 256 elements in the fourth digit. It is clear that the use of curved elements increases the accuracy in the case of curvilinear contours considerably.

Example 2. The use of a terminal element for a flow with a radical singularity. Consider a straight crack in a rectangular plate (Fig. 2). The length of the crack is $2l_C$, it makes an angle with a horizontal side of the plate α_c, the dimensions of the plate are a and b, its conductance is k and the temperature on the crack surface is equal to T_1 and the temperature on the contour of the plate is equal to T_2. Th total influx of heat into the crack ΔJ_C must be equal to the total influx ΔJ_{ext} through the contour of the plate, and the equality $\Delta J_C = \Delta J_{\text{ext}}$ can therefore serve as an additional control of the accuracy of the calculations.

In the problem considered, the discontinuity in the flow on the crack tends to infinity on approaching the tips in accordance with the formula

$$\Delta q_n = k_q/\sqrt{d} \tag{7.1}$$

where k_q is the flow intensity factor (FIF). It follows from the asymptotics (7.1) that it makes sense to use terminal elements with a radical singularity.

We will now compare the accuracy of the calculation of the flow with and without terminal elements. To be specific, we take

$$a = b = 1, \quad 2l_C = 0.75, \quad \alpha_c = 0, \quad k = -1, \quad T_1 = 0, \quad T_2 = 1$$

Calculations were carried out for various numbers of elements on the outer boundary (from 24 to 80), various numbers of regular elements in the crack (from 10 to 140) and for various lengths of the elements close to the crack tips. It was found that an increase in the number of elements on the outer boundary has a small effect on the flow Δq_r at the points of the crack and changing their number from 24 to 80 only affects the fifth significant digit. The size of the elements adjacent to the tips turns out to have the greatest effect. The asymptotics (7.1) only develop at a very short distance d from a tip, of the order of $0.01l_C$. It is only when the size of a terminal element was less than $0.005l_C$ that the calculated values of the FIF retained three unchanged significant digits. In the case of the finest mesh (80 elements on the outer contour, and 150 regular and two terminal elements in the crack), the error in the calculation of the FIF did not exceed one unit in the fourth digit. For the three mesh points 1, 2 and 3 closest to the tip, the distance d to the tip was 0.000667, 0.002 and 0.003333 respectively. The flow Δq_n at these points, calculated using terminal elements was equal to 101.02, 58.32 and 45.18. For comparison, when the terminal elements are replaced by regular elements at the same points, the flow is equal to 130.64, 59.66 and 45.59. Hence, the error when regular elements are used is 29%, 2.3% and 0.9% for mesh points 1, 2 and 3 respectively. At other points of the crack, it does not exceed 0.5%. The total influx into the crack ΔJ_C is equal to 6.302255 and the influx through the contour of the plate agrees with this value up to eight digits.

Example 3. A *Strongly conducting crack in a rectangular plate*. The results presented below were obtained for 56 elements on the contour of the plate, and 140 regular and two terminal elements in the crack (the total number of unknowns was 594). According to the estimate obtained, the error in the calculated values of the FIF does not exceed two units in the fourth digit. The error in the total influx did not exceed 10^{-7} (the influx itself was of the order of unity). The calculated values of the normalized FIF

$$k_{Nq} = \frac{k_q}{|k|(T_2 - T_1)\sqrt{a}}$$

when $a = 1$ and of the normalized influx

$$\Delta J_{nC} = \frac{\Delta J_C}{|k|(T_2 - T_1)}$$

are presented below

$2l_C/a$	0.001	0.01	0.05	0.10	0.15	0.20	0.25	0.50	0.75
k_{Nq}	8.344	3.769	2.379	2.062	1.940	1.884	1.877	1.992	2.608
ΔJ_{nC}	0.8185	1.1692	1.6690	2.0456	2.3568	2.6421	2.9154	4.3289	6.3022

It can be seen that the FIF increases sharply when the crack tips approach the vertical boundaries of the plate. This is explained by the concentration of the flow in the zones between the crack and the boundaries. The existence of a minimum in the dependence of k_{Nq} on $2l_C/a$ is less obvious. This arises on account of the sharp increase in the FIF when the ratio $2l_C/a$ becomes less than 0.1. This increase can be understood if the asymptotic analytical solution

$$k_{Nq} = \frac{2}{\sqrt{2al_C}\ln(a/l_C)}$$

for a small crack in a disc of a diameter a is used when $2l_C/a \to 0$. This solution shows that the FIF tends to infinity when $2l_C/a \to 0$. When $a = 1$ and $2l_C/a = 0.001$, the asymptotic value of the FIF is $k_{Nq} = 8.321$ against the calculated value $k_{Nq} = 8.344$ presented above and, when $2l_C/a = 0.01$, the asymptotic value is equal to $k_{Nq} = 3.774$ against the value presented above $k_{Nq} = 3.769$. The agreement between the results is obvious. For the flow, we have the asymptotic formula

$$\Delta J_{nC} = \frac{2\pi}{\ln(a/l_C)}$$

When $2l_C/a = 0.01$, we have $\Delta J_{nC} = 1.1859$ against $\Delta J_{nC} = 1.1692$, and the agreement is completely satisfactory.

The dependence of the FIF on the angle of orientation of the crack is shown below

α_c	$0, \pi/2$	$\pi/6, \pi/3$	$\pi/4$
$2l_C/a = 0.25$	1.877	1.876	1.875
$2l_C/a = 0.50$	1.992	1.956	1.944
$2l_C/a = 0.75$	2.608	2.321	2.230

It can be seen that this is a weak dependence even in the case of a long crack ($2l_C/a = 0.75$).

Finally, the dependence of the FIF k_{nq} and the flow ΔJ_{nC} on the ratio b/a of the sides of the plate for a horizontal ($\alpha_c = 0$) crack of length $2l_C = 0.75a$ is given below

b/a	0.1	0.2	0.3	0.4	0.5	0.75	1.0
k_{nq}	5.208	3.623	3.158	2.917	2.791	2.656	2.608
ΔJ_{nC}	31.7666	16.8158	11.9580	9.6502	8.3585	6.8637	6.3022

It is obvious that both k_{nq} and ΔJ_{nC} increase as the heated horizontal sides of the plate approach the crack. It is also clear that, for small values of b/a, the total influx becomes roughly inversely proportional to the distance from the crack to the horizontal side of the plate. This might be expected since the flow becomes practically homogeneous in the narrow zones between the crack and the horizontal sides.

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